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Systematic drift experienced by a point vortex in two-dimensional turbulence

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Using a linear response theory, we show that a point vortex in two-dimensional turbulence experiences a systematic drift superposed to its mean-field velocity. Taking this result into account, we derive a Fokker-Planck equation for the evolution of its distribution function and make the link with a maximum entropy production principle [R. Robert and J. Sommeria, Phys. Rev. Lett. **69**, 2776 (1992)]. We also discuss an analogy with stellar systems [P. H. Chavanis, J. Sommeria, and R. Robert, Astrophys. J. **471**, 385 (1996)]; in particular, the *systematic drift* of the vortex is the counterpart of the *dynamical friction* experienced by a star due to close encounters [S. Chandrasekhar, Rev. Mod. Phys. **20** (3) (1949); H. E. Kandrup, Astrophys. Space Sci. **97**, 435 (1983)]. [S1063-651X(98)51507-3]

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It is often useful in two-dimensional turbulence to approximate a continuous field of vorticity by a cloud of point vortices $\omega(\mathbf{r},t)=\Sigma_i\gamma_i\delta(\mathbf{r}-\mathbf{r}_i(t))$ where γ_i is the circulation of vortex i. The main interest is that such a system is described by a Hamiltonian [1] $H=\Sigma_{i<j}\gamma_i\gamma_j W(\mathbf{r}_i,\mathbf{r}_j)$ (where W is the Green function of the domain $\mathcal{D}$) and can be studied by rather ordinary statistical mechanics. This was first considered by Onsager [2], who showed qualitatively the existence of equilibrium states with negative temperatures at which the vortices cluster. He could therefore explain the occurrence of large scale vortices (or ''supervortices'') often observed in Nature. His work was pursued by Joyce and Montgomery [3] in a mean-field approximation. They derived in particular a Maxwell-Boltzmann statistics for the distribution of point vortices at equilibrium.

We are rather interested here in the relaxation towards equilibrium. To that purpose, we wish to derive a stochastic Langevin equation describing the motion of a test vortex traveling in a ''sea'' of field vortices. To a first approximation, the test vortex is driven by the smooth mean-field velocity induced by the rest of the system. It is also subjected to rapid fluctuations arising from the departure to the mean field. Furthermore, we show that it must experience a systematic drift. Indeed, as it travels among the sea of vortices, it alters their distribution; in response, the system exerts a back reaction that modifies its initial trajectory. This is the physical reason for its drift. At equilibrium, the drift balances the scattering and maintains nontrivial density distributions.

There is a strong analogy between two-dimensional (2D) vortices and stellar systems [4]. In this analogy, the systematic drift of a point vortex is the counterpart of the dynamical friction experienced by a star. This dynamic friction has been calculated by Kandrup [5] in a mean-field approach, using a linear response theory. We adapt his procedure here to the case of point vortices.

Consider a collection of N point vortices interacting through the potential W. We shall focus on the situation when the number of point vortices is very large $N\to\infty$ but the total energy remains finite (this implies $\gamma\sim 1/N\to 0$). In this (mean-field) limit, the existence of an equilibrium state is well established [6]. The N-particle distribution function $\mu_{\rm eq}(\{\mathbf{r}_k\})$ can be approximated by a product of N one-particle distribution functions $P_k^{\rm eq}$, each of which at equilibrium [3] with the same inverse temperature $\beta_{\rm eq}$:

$$\mu_{\rm eq}(\{\mathbf{r}_k\})=\prod_{k=1}^{N} P_k^{\rm eq}(\mathbf{r}_k)=\prod_{k=1}^{N} A_k e^{-\beta_{\rm eq}\gamma_k\psi_{\rm eq}(\mathbf{r}_k)}. \tag{1}$$

In the Boltzmann factor, the stream function $\psi_{\rm eq}$, determined in the self-consistent-field approximation, plays the role of an interaction potential. Therefore, the system behaves like an ''ideal'' vortex plasma where the analog of the Debye sphere is the supervortex itself (of size $|\mathcal{D}|$).

The introduction of an additional point vortex (referred to as a ''test vortex'') will modify this equilibrium state. The distribution function becomes

$$\mu(\{\mathbf{r}_k\},t)=\mu_{\rm eq}(\{\mathbf{r}_k\})+\mu'(\{\mathbf{r}_k\},t), \tag{2}$$

where the perturbation $\mu'(\{\mathbf{r}_k\},t)$ reflects the influence of the test vortex on its neighbors (just like in a polarization process). The Hamiltonian of the system can be split in two terms:

$$H=H_{\rm eq}+H_{\rm int}=\sum_{i<j}\gamma_i\gamma_j W(\mathbf{r}_i,\mathbf{r}_j)+\sum_{i=1}^{N}\gamma_i\gamma_0 W(\mathbf{r}_i,\mathbf{r}_0) \tag{3}$$

and the N-particle distribution function $\mu(\{\mathbf{r}_k\},t)$ satisfies the Liouville equation:

$$\frac{\partial\mu}{\partial t}+\sum_{i=1}^{N}\left(\sum_{j\neq i}\gamma_j\mathbf{V}(j\to i)+\gamma_0\mathbf{V}(0\to i)\right)\frac{\partial\mu}{\partial\mathbf{r}_i}=0, \tag{4}$$

where

$$\mathbf{V}(j\to i)=-\mathbf{z}\times\frac{\partial W(\mathbf{r}_j,\mathbf{r}_i)}{\partial\mathbf{r}_i} \tag{5}$$

is the velocity created by a point vortex (of unit circulation) located in $\mathbf{r}_j$ on a point vortex located in $\mathbf{r}_i$. In an infinite domain, $W^s(\mathbf{r}_j,\mathbf{r}_i)=-(1/2\pi)\ln|\mathbf{r}_j-\mathbf{r}_i|$, so that

$$\mathbf{V}^s(j\to i)=-\frac{1}{2\pi}\,\mathbf{z}\times\frac{\mathbf{r}_j-\mathbf{r}_i}{|\mathbf{r}_j-\mathbf{r}_i|^2}$$

diverges like $1/r$ at small distances.

Substituting Eq. (2) into Eq. (4), we obtain the evolution equation of the perturbation μ':

$$\frac{\partial\mu'}{\partial t}+\mathcal{L}\mu'=\beta_{\rm eq}\sum_{i=1}^{N}\mathbf{V}^i\gamma_i\frac{\partial\psi_{\rm eq}}{\partial\mathbf{r}}(\mathbf{r}_i)\mu_{\rm eq}(\{\mathbf{r}_k\}),\tag{6}$$

where

$$\mathcal{L}\equiv\sum_{i=1}^{N}\mathbf{V}^i\frac{\partial}{\partial\mathbf{r}_i}\tag{7}$$

is a Liouville operator and $\mathbf{V}^i=\Sigma_{j\neq i}\gamma_j\mathbf{V}(j\to i)+\gamma_0\mathbf{V}(0\to i)$ denotes the total velocity of vortex i. This equation can be solved formally with the Greenian

$$G(t,t')\equiv\exp\left\{-\int_{t'}^{t}\mathcal{L}(\tau)d\tau\right\}.\tag{8}$$

If $t=0$ is the time at which the test vortex is introduced in the system, we have $\mu'(t=0)=0$. One then finds that

$$\mu'(t)=\beta_{\rm eq}\int_0^t d\tau G(t,t-\tau)\sum_{i=1}^{N}\mathbf{V}^i\gamma_i\frac{\partial\psi_{\rm eq}}{\partial\mathbf{r}}(\mathbf{r}_i)\mu_{\rm eq}(\{\mathbf{r}_k\}).\tag{9}$$

The average velocity of the test vortex is expressed in terms of the distribution function μ of the field vortices by

$$\langle\mathbf{V}^0\rangle=\int\prod_{k=1}^{N}d^2\mathbf{r}_k\mathbf{V}^0\mu(\{\mathbf{r}_k\},t),\tag{10}$$

where $\mathbf{V}^0=\Sigma_{i=1}^{N}\gamma_i\mathbf{V}(i\to 0)$. Substituting the formal result (9) into Eq. (10), one obtains

$$\langle\mathbf{V}^0\rangle=\int\prod_{k=1}^{N}d^2\mathbf{r}_k\mathbf{V}^0\mu_{\rm eq}(\{\mathbf{r}_k\})+\beta_{\rm eq}\int\prod_{k=1}^{N}d^2\mathbf{r}_k\mathbf{V}^0\times\int_0^t d\tau G(t,t-\tau)\times\sum_{i=1}^{N}V_\nu^i\gamma_i\frac{\partial\psi_{\rm eq}}{\partial r^\nu}(\mathbf{r}_i)\mu_{\rm eq}(\{\mathbf{r}_k\}),\tag{11}$$

with summation over repeated greek indices. The two terms arising in this expression have a clear physical meaning. The first term is the mean-field velocity $\langle\mathbf{V}^0\rangle_{\rm eq}=-\mathbf{z}\times\nabla\psi_{\rm eq}(\mathbf{r}_0)$ created by the unperturbed distribution function $\mu_{\rm eq}(\{\mathbf{r}_k\})$. The second term, arising from the perturbation μ', corresponds to the response of the system to the polarization induced by the presence of the test vortex. Because of this back reaction, the test vortex will experience a systematic drift $\langle\mathbf{V}^0\rangle_{\rm drift}=\langle\mathbf{V}^0\rangle-\langle\mathbf{V}^0\rangle_{\rm eq}$. Explicating the action of the Greenian (8), we obtain

$$\langle\mathbf{V}^0\rangle_{\rm drift}=\beta_{\rm eq}\int\prod_{k=1}^{N}d^2\mathbf{r}_k\sum_{i=1}^{N}\gamma_i\mathbf{V}(i\to 0,t)\int_0^t d\tau\times\sum_{i=1}^{N}\left(\sum_{j\neq i}\gamma_jV_\nu(j\to i,t-\tau)+\gamma_0V_\nu(0\to i,t-\tau)\right)\times\gamma_i\frac{\partial\psi_{\rm eq}}{\partial r^\nu}(\mathbf{r}_i(t-\tau))\mu_{\rm eq}(\{\mathbf{r}_k(t-\tau)\}),\tag{12}$$

where $\mathbf{r}_i(t-\tau)$ is the position at time $t-\tau$ of the point vortex i located at $\mathbf{r}_i(t)=\mathbf{r}_i$ at time t. This is obtained by solving the Kirchhoff-Hamilton equations of motion

$$\frac{d\mathbf{r}_i}{dt}=\mathbf{V}_i\tag{13}$$

between t and $t-\tau$.

The *exact* expression of the drift (12) is completely inextricable in the general case. In order to clarify its physical content, we have to make some approximations. We shall consider, in the calculation of the integrals, that the point vortices are purely advected by the mean-field velocity $\langle\mathbf{V}\rangle_{\rm eq}$. This is reasonable because, when $N\to\infty$, the typical velocity fluctuations $\mathcal{V}$, of order $\gamma/d\sim(\gamma/L)N^{1/2}$ (where d is the average distance between two point vortices and L the supervortex size), are much smaller than the mean-field velocity $\langle\mathbf{V}\rangle_{\rm eq}$ of order $N\gamma/L$. Of course, this approximation breaks up at scales smaller than $\delta\sim L/N$ when the velocity fluctuations become comparable to the average velocity. In that case, we cannot ignore the details of the discrete vortex interactions anymore and a specific treatment is necessary. This is, however, beyond the scope of this article. For simplicity, we shall remain in the mean-field approximation (with the aforementioned limitation in mind) and replace the exact Greenian G by a smoother Greenian $\langle G\rangle_{\rm eq}$ constructed with the averaged Liouville operator $\langle\mathcal{L}\rangle_{\rm eq}\equiv\Sigma_{i=1}^{N}\langle\mathbf{V}^i\rangle_{\rm eq}(\partial/\partial\mathbf{r}_i)$. In this approximation the correlations involving two different vortex pairs vanish, so that

$$\langle V_\mu^0\rangle_{\rm drift}=\beta_{\rm eq}\int\prod_{k=1}^{N}d^2\mathbf{r}_k\int_0^t d\tau\sum_{i=1}^{N}\gamma_iV_\mu(i\to 0,t)\times\gamma_0V_\nu(0\to i,t-\tau)\gamma_i\frac{\partial\psi_{\rm eq}}{\partial r_\nu}(\mathbf{r}_i(t-\tau))\prod_{k=1}^{N}P_k^{\rm eq}(\mathbf{r}_k)\tag{14}$$

$[P^{\rm eq}(\mathbf{r}_k(t-\tau))=P^{\rm eq}(\mathbf{r}_k(t))$, since $P_{\rm eq}=f(\psi_{\rm eq})$ is constant over a streamline]. For identical vortices with circulation γ, one obtains

$$\langle V_\mu^0\rangle_{\rm drift}=-N\gamma^3\beta_{\rm eq}\int d^2\mathbf{r}_1\int_0^t d\tau V_\mu^s(1\to 0,t)\times V_\nu^s(1\to 0,t-\tau)\frac{\partial\psi_{\rm eq}}{\partial r_\nu}(\mathbf{r}_0(t-\tau))P^{\rm eq}(\mathbf{r}_0).\tag{15}$$

Since the integral is dominated by the divergence of the product $V_\mu V_\nu$ when $\mathbf{r}_1 \to \mathbf{r}_0$, we have replaced the velocity (5) by its singular part $\mathbf{V}^s$ (neglecting the boundary term) and made the ''local approximation'' $\partial_\nu \psi(\mathbf{r}_1(t-\tau)) \simeq \partial_\nu \psi(\mathbf{r}_0(t-\tau))$ and $P^{\text{eq}}(\mathbf{r}_1) \simeq P^{\text{eq}}(\mathbf{r}_0)$.

The general expression of the drift (15) can be explicated for particular equilibrium flows. If the equilibrium flow is unidirectional, such that $\langle \mathbf{V} \rangle_{\text{eq}} = \langle V \rangle_{\text{eq}}(y)\mathbf{x}$, the trajectory of a point vortex advected by this flow is simply $y(t-\tau) = y(t)$, $x(t-\tau) = x(t) - \langle V \rangle_{\text{eq}}(y)\tau$. Since $\partial \psi_{\text{eq}} / \partial \mathbf{r}[\mathbf{r}_0(t-\tau)] = \partial \psi_{\text{eq}} / \partial y(y_0)\mathbf{y}$, the drift can be written as

$$\langle \mathbf{V}^0 \rangle_{\text{drift}} = -\beta_{\text{eq}} \gamma (D \nabla \psi^0_{\text{eq}} + D_a \langle \mathbf{V}^0 \rangle_{\text{eq}}) \tag{16}$$

where $D = D_{xx} = D_{yy}$ and $D_a = D_{xy} = -D_{yx}$ are the isotropic and anisotropic parts of the diffusion tensor $D_{\mu\nu}$ given by a Kubo formula

$$D_{\mu\nu} = N\gamma^2 \int_0^t d\tau \int d^2\mathbf{r}_1 V^s_\mu(1 \to 0,t) V^s_\nu(1 \to 0,t-\tau) P_{\text{eq}}(\mathbf{r}_0). \tag{17}$$

These coefficients can be calculated explicitly. For example,

$$D_{yy} = \frac{N\gamma^2}{4\pi^2} \int_0^t d\tau \int dx_1 dy_1 \frac{x_1 - x_0}{(x_1-x_0)^2 + (y_1-y_0)^2} (t) \times \frac{x_1 - x_0}{(x_1-x_0)^2 + (y_1-y_0)^2} (t-\tau) P_{\text{eq}}(y_0). \tag{18}$$

Since the integral is dominated by close interactions, we can make the approximation

$$\langle V \rangle_{\text{eq}}(y_1) - \langle V \rangle_{\text{eq}}(y_0) \simeq -\Sigma(y_0)(y_1 - y_0) \tag{19}$$

where $\Sigma \equiv -\partial \langle V \rangle_{\text{eq}} / \partial y$ is the local shear (equal here to the vorticity). Introducing the variables $X \equiv x_1 - x_0$, $Y \equiv y_1 - y_0$ we obtain

$$D_{yy} = \frac{N\gamma^2}{4\pi^2} P_{\text{eq}}(y_0) \int_0^t d\tau \int dX\, dY \frac{X}{X^2+Y^2} \times \frac{X + \Sigma Y \tau}{(X+\Sigma Y\tau)^2 + Y^2}. \tag{20}$$

The integrations over X and τ can be performed easily, leading to

$$D = \frac{N\gamma^2}{2\pi|\Sigma|} \arctan\left(\frac{|\Sigma|}{2} t \right) \ln \Lambda P_{\text{eq}}(y_0), \tag{21}$$

where $\ln \Lambda \equiv \int_0^{+\infty} dY/Y$. This integral diverges logarithmically for both small and large Y. There is a similar divergence in plasma physics and for stellar systems due to the long range nature (and the singularity for $r \to 0$) of a potential in r^{-1} or, here, $\ln r$. As mentioned previously, the divergence at small Y accounts for the failure of the mean-field approximation on scale δ. This is the main limitation of our theory. This problem could be resolved in principle by a more precise modeling of the discrete vortex interactions (in the spirit of a kinetic model). This would amount to a regularization of the $1/Y$ integrant at scale $\sim\delta$. We shall circumvent this difficulty by introducing a cutoff at that scale: $Y_{\text{min}} \sim \delta$. The divergence at large Y is solved by the finite extent of the system; in plasma physics, we would stop the integration at λ_D (the Debye length), but in our case the interaction is unshielded (except in the geophysical case, where the Rossby radius plays the same role as the Debye length). It is therefore natural to cut the integral at $Y_{\text{max}} \sim L$, the vortex size. We shall take accordingly $\ln \Lambda = \ln(L/\delta) \sim \ln N$. Since the divergence is weak (logarithmic), the result does not depend too much on the precise value of the cutoffs.

A similar calculation gives

$$D_a = -\frac{N\gamma^2}{4\pi\Sigma} \ln\left(1 + \frac{\Sigma^2 t^2}{4} \right) \ln \Lambda P_{\text{eq}}(y_0). \tag{22}$$

Formulas (16), (21), and (22) remain valid in the case of an axisymmetric equilibrium flow, now with $\Sigma = r(d/dr)(\langle V_\theta \rangle / r)$. They also remain valid in the general case, if $|\nabla \psi_{\text{eq}}|$ does not vary too much along a streamline (Σ is then replaced by $\langle \Sigma \rangle$, the average shear over a streamline). The drift (16) has two components: the component $-\beta_{\text{eq}} \gamma D \nabla \psi^0_{\text{eq}}$ associated with the isotropic diffusion and the component $-\beta_{\text{eq}} \gamma D_a \langle \mathbf{V}^0 \rangle_{\text{eq}}$ due to anisotropic effects. These anisotropic effects are somewhat secondary since they introduce a component *parallel* to the mean flow. We shall therefore keep only the component *perpendicular* to the mean-field velocity, as it is responsible for a real deviation of the vortex:

$$\langle \mathbf{V}^0 \rangle_{\text{drift}} = -\beta_{\text{eq}} \gamma D \boldsymbol{\nabla} \psi^0_{\text{eq}}. \tag{23}$$

This drift has the same physical origin as the dynamical friction $\langle \mathbf{F}^0_{\text{fr}} \rangle = -D \beta_{\text{eq}} m \mathbf{v}$ experienced by a star, due to close encounters [5]. The expression of the drift coefficient $\xi = \beta_{\text{eq}} \gamma D$ is an amusing generalization of Einstein's formula to the case of point vortices (ξ corresponds to the ordinary friction coefficient of colloidal particles or stars [7]). The direction of the drift has important physical implications. Consider a point vortex moving at the periphery of the system. Its motion is anticlockwise if we assume positive circulation. For negative temperatures, the drift is directed to its left: the vortex is *attracted* to the center of the domain. On the contrary, for positive temperatures, the drift is directed to its right: the vortex is *rejected* toward the boundary. This reflects the general structure of the equilibrium state [2] and gives a physical mechanism for the organization of point vortices at negative temperatures.

We can try now to derive an evolution equation for the probability $P(\mathbf{r}_0, t)$ of finding the test vortex in $\mathbf{r}_0$ at time t. Having evidenced the existence of a systematic drift, we can write down a stochastic Langevin equation for the motion of the vortex:

$$\Delta \mathbf{r}_0 = \langle \mathbf{V}^0 \rangle_{\text{eq}} \Delta t - \xi \boldsymbol{\nabla} \psi^0_{\text{eq}} \Delta t + \mathbf{B}(\Delta t). \tag{24}$$

The first term corresponds to the mean-field velocity, the second to the drift, and the third to fluctuations arising from the difference between the exact distribution of the vortices $\{\mathbf{r}_i\}$ and their ''smoothed out'' distribution $P_{\text{eq}}(\mathbf{r}_i)$. We can now apply the standard techniques of Brownian theory [7] (the diffusion approximation is well justified since the fluctuations $\mathcal{V}\ll\langle\mathbf{V}\rangle_{\rm eq}$ produce a *large* number of *weak* displacements). Assuming that the motion of the point vortex is Markovian and that the fluctuations $\mathbf{B}(\Delta t)$ can be described by a Gaussian stochastic process, we obtain the Fokker-Planck equation

$$\frac{\partial P_0}{\partial t}+\langle\mathbf{V}^0\rangle_{\rm eq}\nabla P_0=\nabla(D\nabla P_0+\xi P_0\nabla\psi^0_{\rm eq}). \qquad (25)$$

The right-hand side is a sum of two terms: the first term is a diffusion due to the erratic motion of the test vortex caused by the fluctuations; the second term corresponds to the drift. Since $\xi=D\gamma\beta_{\rm eq}$, we find that the test vortex will ultimately relax towards the equilibrium distribution $P_{\rm eq}$ of the field vortices.

If we are not too far from equilibrium, we can try to apply this equation to the evolution of the flow itself. We are led therefore to introduce the average vorticity $\langle\omega\rangle=N\gamma P(\mathbf{r},t)$ and replace the equilibrium field $\psi_{\rm eq}$ by the field ψ produced by $\langle w\rangle$. We therefore obtain the coupled system

$$\frac{\partial\langle\omega\rangle}{\partial t}+\langle\mathbf{V}\rangle\nabla\langle\omega\rangle=\nabla[D(\nabla\langle\omega\rangle+\beta(t)\gamma\langle\omega\rangle\nabla\psi)], \qquad (26)$$

$$\langle\omega\rangle=-\Delta\psi. \qquad (27)$$

The inverse temperature is now a function of time determined by the conservation of energy: $\beta(t)=-\int D\nabla\langle\omega\rangle\nabla\psi d^2\mathbf{r}/\int D\langle\omega\rangle\gamma(\nabla\psi)^2d^2\mathbf{r}$. Equation (26) is consistent with a maximum entropy production principle (MEPP) originally introduced in the case of continuous vorticity fields [8] (its application to the case of point vortices is straightforward). This principle (which can be viewed as a variational version of linear thermodynamics) capitalizes on one's ignorance and assumes that ''during its evolution, a system tends to maximize its rate of entropy production while satisfying all the constraints imposed by the dynamics'' (this is a clear extension of the well known principle of equilibrium thermodynamics). In this framework, the diffusion term results from the variations of the entropy while the drift term is necessary to conserve energy (the Einstein relation is automatically satisfied by this variational principle). However, the MEPP does not give the value of the diffusion coefficient that appears as an ill-defined Lagrange multiplier. By contrast, our model provides an explicit expression,

$$D=\frac{\gamma\tau}{8\pi}\ln\Lambda\langle\omega\rangle, \qquad (28)$$

where $\tau\sim2\pi/\langle\Sigma\rangle$, according to Eq. (21). This shows that the time of correlation is not short, but of order $t_D=\langle\omega\rangle^{-1}$, the dynamical time. Therefore, anisotropic effects (due to memory terms) may be important. In that case, the diffusion current in Eq. (26) is replaced by $D\mathbf{W}-D_a\mathbf{z}\times\mathbf{W}$ where $\mathbf{W}=\nabla\langle\omega\rangle+\beta(t)\gamma\langle\omega\rangle\nabla\psi$. However, the physical relevance of the diffusion current $-D_a\mathbf{z}\times\mathbf{W}$ is questionable, since it acts along the streamlines and does not change the entropy $S=-\int d^2\mathbf{r}\langle\omega\rangle\ln\langle\omega\rangle$.

It must be kept in mind that the point vortex model is a crude approximation of real flows (with continuous vorticity). The statistical mechanics of continuous vorticity fields has been considered by several authors (see references in [4]). The averaging procedure refers to a ''coarse-graining'' of the vorticity field and the equilibrium state belongs to the Fermi-Dirac statistics. A relaxation equation can be obtained from the MEPP [8,9] and is similar to Eq. (26) with, however, two important differences: (i) the drift is nonlinear in $\overline{\omega}$; (ii) the diffusion coefficient is much larger, accounting for a more ''violent'' relaxation [in the case of point vortices, the relaxation time $t_{\rm pv}\sim(N/\ln N)t_D$, estimated from Eq. (28), can be very long]. Except for these two (important) differences, the relaxation equations are morphologically similar. This may increase interest in the point vortex model.

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[1] G. Kirchhoff, *Vorlesungen uber matematische Physik* (B. G. Teubener, Leipzig, 1877).
[2] L. Onsager, Nuovo Cimento Suppl. **6**, 229 (1949).
[3] G. Joyce and D. Montgomery, J. Plasma Phys. **10**, 107 (1973).
[4] P. H. Chavanis, J. Sommeria, and R. Robert, Astrophys. J. **471**, 385 (1996).
[5] H. E. Kandrup, Astrophys. Space Sci. **97**, 435 (1983).
[6] G. L. Eyink and H. Spohn, J. Stat. Phys. **70**, 833 (1993).
[7] S. Chandrasekhar, Rev. Mod. Phys. **20**(3), 1 (1949).
[8] R. Robert and J. Sommeria, Phys. Rev. Lett. **69**, 2776 (1992).
[9] R. Robert and C. Rosier, J. Stat. Phys. **86**, 481 (1997).